

Gabor frames with rational density.

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Abstract

We consider the frame property of the Gabor system $\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i \beta n t} g(t - \alpha m) : m, n \in \mathbb{Z}\}$ for the case of rational oversampling, i.e. $\alpha, \beta \in \mathbb{Q}$. A 'rational' analogue of the Ron-Shen Gramian is constructed, and prove that for any odd window function g the system $\mathcal{G}(g, \alpha, \beta)$ does not generate a frame if $\alpha\beta = \frac{n-1}{n}$. Special attention is paid to the first Hermite function $h_1(t) = te^{-\pi t^2}$.

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1 Introduction

One of the fundamental problems of Gabor analysis can be stated as follows: given a window function $g \in L^2(\mathbb{R})$, determine the set of lattice parameters $\alpha, \beta > 0$ such that the Gabor system $\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i \beta n t} g(t - \alpha m) : m, n \in \mathbb{Z}\}$ forms a frame in $L^2(\mathbb{R})$.

$$\mathcal{F}(g) := \{(\alpha, \beta) \in \mathbb{R}_+^2 : \mathcal{G}(g, \alpha, \beta) \text{ is a frame for } L^2(\mathbb{R})\}.$$

We remind some known facts about the frame set $\mathcal{F}(g)$, following [4] and (in more compressed form) [6]. Under milder conditions, precisely if g is in the Feichtinger algebra M^1 the set $\mathcal{F}(g)$ is open in \mathbb{R}_+^2 and contains a neighborhood of the origin. Fundamental density theorems [3, 4, 7] together with a version of the uncertainty principle [1, 2] asserts that also $\mathcal{F}(g) \subset \Pi_+ := \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$ for $g \in M^1$.

Up to the very recent time just few functions have been known for which $\mathcal{F} = \Pi_+$. The list included the Gaussian $g(t) = e^{-\pi t^2}$ [11, 13, 14], the hyperbolic secant $g(t) = (e^t + e^{-t})^{-1}$, one- and two-sided exponential functions $g(t) = e^{-t} \mathbf{1}_{\mathbb{R}_+}$ (in this case $\alpha\beta = 1$ also generates a frame) and $g(t) = e^{-|t|}$ [8, 10], as well as their shifts, dilates, and Fourier transforms. A breakthrough was achieved in [6] where the authors constructed an infinite family of functions for which $\mathcal{F}(g) = \Pi_+$ by proving that any totally positive function of finite type possesses this property.

On the other hand it is shown in [9] that the set $\mathcal{F}(g)$ may have rather complicated structure even for the "simple" function such as the characteristic function $g = \mathbf{1}_I$ of an interval.

In this article we attempt to study the set $\mathcal{F}(g)$ for some cases when $\mathcal{F}(g) \neq \Pi_+$ and g is well concentrated both in time and frequency. Our primary objective is the first Hermite function $h_1(t) = te^{-\pi t^2}$. This choice is motivated by the uncertainty principle (h_1 minimizes the Heisenberg uncertainty among all functions which are orthogonal to the Gaussian) and also recent results regarding vector-valued Gabor frames [5].

The article is organized as follows. The next section contains notation and basic facts from Gabor analysis which will be used in the sequel. The whole analysis is carried out for the case of rational oversampling: $\alpha\beta = p/q \in \mathbb{Q}$. In Section 3 we prove that for *any* odd function $g \in M^1$ (in particular h_1 of course) the set $\mathcal{F}(g)$ does NOT contain the union of hyperbolas:

$$\alpha\beta = \frac{n-1}{n} \Rightarrow (\alpha, \beta) \notin \mathcal{F}(g), \quad n = 2, 3, \dots \quad (1)$$

The proof is based on analysis of the vector-valued Zak transform (see [16] and also [4, ch. 8]) which represents the frame operator as matrix multiplication in a space of vector-valued functions. In the next section we factorize the matrix of the vector-valued Zak transform and extract a factor which is a rational analogue of the well-known Ron-Shen Gramian (see [12] and also [4]). In Section 5 we conjecture that condition (1) is the only restriction on the set $\mathcal{F}(h_1)$:

$$\mathcal{F}(h_1) = \{(\alpha, \beta) \in \Pi_+ : \alpha\beta \neq \frac{n-1}{n}, \quad n = 2, 3, \dots \}.$$

Unfortunately we are not able to prove this conjecture in its full range. We prove it analytically just for some points in Π_+ and also provide numerical verification for a wider set of points. These constructions are based on the rational analogue of the Gramian given in Section 4.

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2 Preliminaries

In this section we remind the basic facts from Gabor analysis which will be used later. We refer the reader to [4] for a more detailed presentation as well as the history of the subject.

Given numbers $\alpha, \beta > 0$ and a window function g we consider the lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and the Gabor system

$$\mathcal{G}(g, \alpha, \beta) = \{\pi_\lambda g : \lambda \in \Lambda\},$$

where $\pi_\lambda : g \rightarrow e^{2\pi i b t} g(t - a)$ for $\lambda = (a, b)$ denotes the usual time-frequency shift. The Gabor frame operator $S_{g, \Lambda} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined as

$$S_{g, \Lambda} f(t) = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle_{L^2(\mathbb{R})} \pi_\lambda g(t), \quad f \in L^2(\mathbb{R}).$$

If g belongs to the modulation space $M^1(\mathbb{R})$ (we remind the definition later in this section) this operator is bounded, and $\mathcal{G}(g, \Lambda)$ is a Gabor frame if and only if the Gabor frame operator is invertible. In this article we consider the case $\alpha\beta \in \mathbb{Q}$. The operator $S_{g, \Lambda}$ can in this case be realized as a multiplication-operator in a space of vector-valued functions.

Let $\alpha\beta = p/q$ for some relatively primes $p, q \in \mathbb{N}$. Consider the rectangle $Q_{\alpha, \beta} = [0, \alpha/p) \times [0, 1/\alpha)$ and the space of vector-valued functions $\mathcal{H}_{\alpha, p} = L^2(Q_{\alpha, p}, \mathbb{C}^p)$.

We remind that the Zak transform is defined as

$$\mathcal{Z}_\alpha f(t, \omega) = \sum_{n \in \mathbb{Z}} f(t - \alpha n) e^{2\pi i n \alpha \omega}. \quad (2)$$

Following [4, ch. 8] we consider the *vector-valued* Zak transform $\vec{\mathcal{Z}}_\alpha : L^2(\mathbb{R}) \rightarrow \mathcal{H}_{\alpha, p}$ defined as

$$\vec{\mathcal{Z}}_\alpha f(x, \omega) = (\mathcal{Z}_\alpha(x + \frac{\alpha}{p}r, \omega))_{r=1}^p, \quad (x, \omega) \in Q_{\alpha, p}$$

The vector-valued Zak transform is up to normalization a unitary mapping between $L^2(\mathbb{R})$ and $\mathcal{H}_{\alpha, p}$.

Denote also

$$A_r^s(x, \omega) = \alpha \sum_{j=0}^{q-1} \overline{\mathcal{Z}_\alpha g(x + \frac{\alpha}{p}s, \omega - \beta j)} \mathcal{Z}_\alpha g(x + \frac{\alpha}{p}r, \omega - \beta j) e^{2\pi i j(r-s)/q},$$

and consider the $p \times p$ matrix function

$$\mathcal{A}(x, \omega) = (A_r^s(x, \omega))_{r, s=0}^{p-1}, \quad (x, \omega) \in Q_{\alpha, p}.$$

Theorem A (Zibulskii-Zeevi) (see [16] and also Theorem 8.3.3. in [4]).
With the above assumptions we have

$$\vec{\mathcal{Z}}_\alpha(S_{g, \alpha, \beta} f)(x, \omega) = \mathcal{A}(x, \omega) \vec{\mathcal{Z}}_\alpha f(x, \omega),$$

for almost all $(x, \omega) \in Q_{\alpha, p}$.

In what follows we assume that g belongs to the modulation space $M^1(\mathbb{R})$.

Definition 1. The modulation space $M^1(\mathbb{R})$ consists functions g for which the norm

$$\|g\|_{M^1(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} |V_f g(x, \omega)| dx d\omega < \infty$$

for some (or equivalently all) non-trivial function f in the Schwartz space $\mathcal{S}(\mathbb{R})$.

If $g \in M^1(\mathbb{R})$ then $\mathcal{Z}_\alpha g$ is continuous.

Corollary 1. The Gabor system $\mathcal{G}(g, \Lambda)$ is a frame in $L^2(\mathbb{R})$ if and only if

$$\det \mathcal{A}(x, \omega) \neq 0, \quad (x, \omega) \in Q_{\alpha, p}. \quad (3)$$

We factorizes the matrix \mathcal{A} in order to make condition (3) more transparent. Consider the column vectors

$$X^j(x, \omega) = (X_r^j(x, \omega))_{r=0}^{p-1}, \quad j = 0, 1, \dots, q-1$$

where

$$X_r^j(x, \omega) = \mathcal{Z}_\alpha g(t + \frac{\alpha r}{p}, \omega - \beta j) e^{2\pi i j r / q}, \quad (4)$$

and the $p \times q$ matrix

$$\mathcal{Q}(t, \omega) = (X^j)_{j=0}^{q-1} = \left(\mathcal{Z}_\alpha g(t + \frac{\alpha r}{p}, \omega - \beta j) e^{2\pi i j r / q} \right)_{r=0, j=0}^{p-1, q-1}.$$

Let \mathcal{Q}^T denote the conjugate transform of \mathcal{Q} . Clearly

$$\mathcal{A}(x, \omega) = \mathcal{Q}(x, \omega) \mathcal{Q}^T(x, \omega).$$

We have

$$\mathcal{A}(x, \omega) \vec{\mathcal{Z}}_\alpha f(x, \omega) = \sum_{j=0}^{q-1} \langle X^j(x, \omega), \vec{\mathcal{Z}}_\alpha f(x, \omega) \rangle \langle X^j(x, \omega), \vec{\mathcal{Z}}_\alpha f(x, \omega) \rangle,$$

so the condition (3) is met if and only if for each $(x, \omega) \in Q_{\alpha, p}$ the vectors $X^j(x, \omega)$, $j = 0, 1, \dots, q-1$ span \mathbb{C}^p .

Corollary 2. Let $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$ and $g \in M^1(\mathbb{R})$. If $\mathcal{G}(g, \Lambda)$ is a frame in $L^2(\mathbb{R})$ it is necessary and sufficient that

$$\text{rank} \mathcal{Q}(x, \omega) = p, \quad \text{for all } (x, \omega) \in Q_{\alpha, p}.$$

In the next section we use this condition in order to study Gabor systems generated by odd functions.

3 Gabor frames generated by odd functions

In this section we prove the following

Theorem 1. *Let $g \in M^1(\mathbb{R})$ be an odd function and $\alpha\beta = \frac{n-1}{n}$, $n = 2, 3, \dots$. Then $\mathcal{G}(g, \Lambda)$ cannot form a frame in $L^2(\mathbb{R})$.*

Proof. We will prove that

$$\text{rank } \mathcal{Q}(0, 0) < n - 1. \quad (5)$$

The results then follows from Corollary 2.

Relation (5) will follow from the fact that for odd windows the elements of the matrix $\mathcal{Q}(0, 0)$ posses additional symmetries. For simplicity we will assume $\alpha = 1$.

Lemma 1. *Let $g \in M^1(\mathbb{R})$ be an odd function, and let $\alpha = 1$, $\beta = p/q \in \mathbb{Q}$ and $X_s^j = X_s^j(0, 0)$, where the functions $X_s^j(x, \omega)$ are defined by (4). Then*

$$X_s^j = -X_{p-s}^{q-j}, \quad s = 0, 1, \dots, p-1, \quad j = 0, 1, \dots, q-1. \quad (6)$$

The proof of the lemma follows readily from the definition of the Zak transform and also from the fact that g is odd. We will use this lemma for $q - p = 1$.

First we consider the case $q = 2k + 2$, $p = 2k + 1$ (q is an even number). In this case $\mathcal{Q}(0, 0)$ is a $(2k + 1) \times (2k + 2)$ matrix.

We need additional relations for the elements of the zero row, the zero column and also the $(k + 1)$ -th column of $\mathcal{Q}(0, 0)$. Namely

$$X_0^0 = 0, \quad X_0^{k+1} = 0, \quad X_0^j = -X_0^{q-j} \quad - \text{zero row} \quad (7)$$

$$X_s^0 = -X_{p-s}^0, \quad s = 1, \dots, p-1 \quad - \text{zero column}; \quad (8)$$

$$X_s^{k+1} = -X_{p-s}^{k+1}, \quad s = 1, \dots, p-1 \quad - (k+1)\text{th column}. \quad (9)$$

As in Lemma 1 these relations follow readily from the definition of the Zak transform.

Let R_s denote the s -th row of $\mathcal{Q}(0, 0)$. Consider the row vectors $e_l = (e_l^j)_{j=0,1,\dots,2k+2}$, $l = 1, 2, \dots, k$, where $e_l^j = 0$, for $j \neq l + 1, 2k + 2 - l$, $e_l^{l+1} = 1$, and $e_l^{2k+2-l} = -1$.

From the relations (6), (7), (8), and (9) it is easy to see that all rows of \mathcal{Q} belong to $\mathcal{S} = \text{span} \{ \{R_s\}_{s=1}^k \} \cup \{e_l\}_{l=0}^k \}$.

Indeed the row R_0 has the form

$$R = (0, \alpha_1, \dots, \alpha_k, 0, -\alpha_k, \dots, -\alpha_1) \quad (10)$$

for some $\alpha_1, \dots, \alpha_k$, thus the vector can be spanned by $\{e_l\}_{l=0}^n$. The rows R_s $s = 1, \dots, k$ belong to the spanning set themselves. So it suffices to prove that the rows R_{p-s} , $s = 1, \dots, k$ also belong to \mathcal{S} or equivalently the vectors $R_s + R_{p-s}$ belong to \mathcal{S} . The later is evident since, according (6) and (8), these vectors also have the form (10).

This completes the proof of the Theorem in the case $q = 2k + 2$, $p = 2k + 1$.

Consider now the case $p = 2k$, $q = 2k + 1$ (q is an odd number). Once again, in addition to the general relation (6), we need relations for selected rows and columns:

$$\begin{aligned} X_0^j &= -X_0^{q-j}, & - \text{zero row;} \\ X_s^0 &= -X_{p-s}^0 & - \text{zero column;} \\ X_k^j &= -X_k^{q-j} & - k\text{-th row.} \end{aligned}$$

Consider now the rows $e_l = (e_l^j)_{j=0,1,\dots,2k}$, $l = 1, 2, \dots, k$ with $e_l^j = 0$ if $j \neq l, 2k - l + 1$, $e_l^j = 1$, and $e_l^{2k-l+1} = -1$.

Applying the same arguments as in the previous case we can see that the set of $2k - 1$ vectors $\{R_s\}_{s=1,\dots,k-1} \cup \{e_l\}_{l=1,\dots,k}$ spans all rows of the matrix $\mathcal{Q}(0, 0)$.

This completes the proof of Theorem 1. \square

4 Factorization of the Zibulskii-Zeevi matrix

In this section we study the Zibulskii-Zeevi matrix $\mathcal{Q}(x, \omega)$. Our goal is to reduce it to a simpler $p \times q$ matrix having the same rank as \mathcal{Q} . One can consider this simpler matrix as an analogue of the Ron-Shen Gramian [12] for rationally oversampled Gabor systems.

We believe this reduction is interesting by itself, it will be also used in the next section in order to study Gabor frames generated by the first Hermite function.

Theorem 2. *Let the window function g belong to $M^1(\mathbb{R})$ and $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. The system $\mathcal{G}(g, \alpha, \beta)$ forms a frame in $L^2(\mathbb{R})$ if and only if the matrix*

$$\mathcal{P}(x, \omega) = \left(Z_{\alpha q} g\left(x + \frac{\alpha}{p}(tp + sq), \omega\right) \right)_{s=0}^{p-1} \quad_{t=0}^{q-1} \quad (11)$$

has rank $= p$ for all $(x, \omega) \in Q_{\alpha, p}$.

Proof. Fix $(x, \omega) \in \mathcal{Q}_{\alpha, p}$ and let

$$X_s^j = Z_{\alpha} g(x + \frac{\alpha}{p}s, \omega - \beta j) e^{2i\pi \frac{js}{q}} = \sum_n g(x + \frac{\alpha}{p}(s - pn)) e^{2i\pi \alpha n \omega} e^{2i\pi j(\frac{s}{q} - \alpha \beta n)} \quad (12)$$

be the corresponding element of the matrix $\mathcal{Q}(x, \omega)$.

For each $s = 0, 1, \dots, p-1$, $t = 0, 1, \dots, q-1$ let

$$L(s) := \{l : l = s - pn, n \in \mathbb{Z}\}, \quad L(s, t) = \{l \in L(s) : l = t + mq, m \in \mathbb{Z}\}.$$

Setting $l = s - pn$ in (12) we obtain

$$\begin{aligned} X_s^j &= \sum_{l \in L(s)} g(x + \frac{\alpha}{p}l) e^{2i\pi \frac{il}{q} + 2i\pi \alpha \omega \frac{s-l}{p}} \\ &= \sum_{t=0}^{q-1} e^{2i\pi \alpha \omega \frac{s}{p}} \sum_{l \in L(s, t)} g(x + \frac{\alpha}{p}l) e^{-2i\pi \alpha \omega \frac{l}{p}} e^{2i\pi \frac{jt}{q}}. \end{aligned} \quad (13)$$

For each $t \in \{0, \dots, q-1\}$ and $s \in \{0, \dots, p-1\}$ we chose numbers $k_t \in \{0, \dots, q-1\}$ and $m_s \in \{0, \dots, p-1\}$ such that

$$k_t p = t \pmod{q}, \quad m_s q = s \pmod{p}.$$

One can easily see that $L(s, t) = \{k_t p + m_s q - pqm : m \in \mathbb{Z}\}$, so one can rewrite (13) as

$$\begin{aligned} X_s^j &= e^{2i\pi \alpha \omega \frac{s - m_s q}{p}} \sum_{t=0}^{q-1} e^{2i\pi \alpha \omega k_t} e^{2i\pi k_t j \frac{p}{q}} \times \\ &\quad \underbrace{\sum_{m \in \mathbb{Z}} g\left(x + \frac{\alpha}{p}(k_t p + m_s q) - m \alpha q\right) e^{2i\pi \omega m \alpha q}}_{= Z_{\alpha q} g(x + \frac{\alpha}{p}(k_t p + m_s q), \omega)}. \end{aligned} \quad (14)$$

Since the numbers k_t runs through the set $\{0, \dots, q-1\}$ as t runs through this set, we can rewrite (14) as

$$\begin{aligned} X_s^j &= e^{2i\pi \alpha \omega \frac{s - m_s q}{p}} \sum_{\tau=0}^{q-1} e^{2i\pi \alpha \omega \tau} e^{2i\pi \tau j \frac{p}{q}} \times \\ &\quad \underbrace{\sum_{m \in \mathbb{Z}} g\left(x + \frac{\alpha}{p}(\tau p + m_s q) - m \alpha q\right) e^{2i\pi \omega m \alpha q}}_{= Z_{\alpha q} g(x + \frac{\alpha}{p}(\tau p + m_s q), \omega)}, \end{aligned}$$

or

$$\mathcal{Q}(x, \omega) = \text{diag}\{e^{2i\pi\alpha\omega \frac{s-m_s q}{p}}\}_{s=0}^{p-1} \tilde{\mathcal{P}}(x, \omega) \text{diag}\{e^{2i\pi\alpha\omega\tau}\}_{\tau=0}^{q-1} W,$$

here

$$\tilde{\mathcal{P}}(x, \omega) = \left(Z_{\alpha q} g\left(x + \frac{\alpha}{p}(\tau p + m_s q), \omega\right) \right)_{s=0}^{p-1}{}_{\tau=0}^{q-1}, \quad W = \left(e^{2i\pi\tau j \frac{p}{q}} \right)_{\tau, j=0}^{q-1}.$$

Clearly the matrices $\tilde{\mathcal{P}}(x, \omega)$ and $\mathcal{Q}(x, \omega)$ have the same rank. On the other hand, since the number m_s runs through the whole set $\{0, 1, \dots, p-1\}$ as s runs through this set the matrices $\tilde{\mathcal{P}}(x, \omega)$ and $\mathcal{P}(x, \omega)$ differ only by permutations of their rows and hence have the same rank. \square

5 Example

In [5] it is proved that the Gabor system $\mathcal{G}(h_1, \alpha, \beta)$ is a frame if $\alpha\beta < \frac{1}{2}$, and fails to be a frame if $\alpha\beta = \frac{1}{2}$. Furthermore the authors discuss an example suggesting that this result may be sharp.

In this section we prove that for at least some α, β with $\alpha\beta > \frac{1}{2}$ the system $\mathcal{G}(h_1, \alpha, \beta)$ indeed generates a frame in $L^2(\mathbb{R})$. The proof uses the matrix-function \mathcal{P} constructed in the previous section, and also a result on *diagonally-dominant* matrices.

Theorem 3. *Let $\alpha\beta = 3/5$ and $h_1(t) = te^{-\pi t^2}$. Then the system $\mathcal{G}(h_1, \alpha, \beta)$ is a frame in $L^2(\mathbb{R})$.*

Proof. The matrix $\mathcal{P}(x, \omega)$ takes the form

$$\mathcal{P}_1(x, \omega) = \left(\mathcal{Z}_{5\alpha} h_1\left(x + \alpha t + s \frac{5\alpha}{3}, \omega\right) \right)_{s, t=0}^{2, 4},$$

By Theorem 2 and Corollary 2 it suffices to prove that

$$\text{rank } \mathcal{P}_1(x, \omega) = 3, \quad \text{for all } (x, \omega) \in Q_{\alpha, 3}.$$

We split the proof into several steps.

- a.** It suffices to prove that $\mathcal{G}(h_1, \alpha, \beta)$, $\alpha\beta = \frac{3}{5}$ is a frame for $\alpha \geq \sqrt{\frac{3}{5}}$. The case $\beta \geq \sqrt{\frac{3}{5}}$ can be reduced to the previous by using the Fourier transform.
- b.** Since the function h_1 decays fast one can approximate $\mathcal{Z}_{5\alpha} h_1(x, \omega)$ with the maximal term of the series. In particular the following holds

Lemma 2. *Let $0 \leq |x| < \frac{5\alpha}{2}$. Then*

$$|h_1(x) - \mathcal{Z}_{5\alpha} h_1(x, \omega)| \leq C_{5\alpha} h_1(5\alpha - |x|)$$

where $C_{5\alpha} = 2 + \frac{1}{h(5\alpha)} \sum_{n \geq 2} h_1(5\alpha n) + h_1(\frac{5\alpha(2n-1)}{2})$.

This lemma can be verified directly. For all practical reasons we can assume that $C_{5\alpha} = 0$.

c. We will see later in **(g)** that it suffices to consider $0 \leq x \leq \frac{\alpha}{6}$. This will follow from symmetry of h_1 and quasi-periodicity of the Zak transform. We split the interval $0 \leq x \leq \frac{\alpha}{6}$ in two: $0 \leq x < \frac{\alpha}{12}$ and $\frac{1}{12} \leq x \leq \frac{\alpha}{6}$.

d. Let $0 \leq x < \frac{\alpha}{12}$. Consider the sub-matrix of $\mathcal{P}_1(x, \omega)$ corresponding to $t = 1, 2, 3$. After interchanging of the second and third row this matrix takes the form

$$\begin{pmatrix} \mathcal{Z}_{5\alpha} h_1(x + \alpha, \omega) & \mathcal{Z}_{5\alpha} h_1(x + 2\alpha, \omega) & \mathcal{Z}_{5\alpha} h_1(x + 3\alpha, \omega) \\ \mathcal{Z}_{5\alpha} h_1(x + \frac{13\alpha}{3}, \omega) & \mathcal{Z}_{5\alpha} h_1(x + \frac{16\alpha}{3}, \omega) & \mathcal{Z}_{5\alpha} h_1(x + \frac{19\alpha}{3}, \omega) \\ \mathcal{Z}_{5\alpha} h_1(x + \frac{8\alpha}{3}, \omega) & \mathcal{Z}_{5\alpha} h_1(x + \frac{11\alpha}{3}, \omega) & \mathcal{Z}_{5\alpha} h_1(x + \frac{14\alpha}{3}, \omega) \end{pmatrix}.$$

We will use the following theorem about diagonally dominant matrices.

Theorem B (see [15]). *If (a_i^k) is an $n \times n$ -matrix with complex elements such that either*

$$(i): \quad |a_i^i| > \sum_{k, k \neq i} |a_k^i|, \quad 1 \leq i \leq n$$

or

$$(ii): \quad |a_i^i| |a_j^j| > \left(\sum_{k, k \neq i} |a_k^i| \right) \left(\sum_{k, k \neq j} |a_k^j| \right), \quad 1 \leq i, j \leq n, i \neq j$$

then $\det(a_{ik}) \neq 0$.

e. We will check that Theorem B can be used to prove invertibility of the matrix in **(d)**. We emphasis the main steps of the proof.

- Using the fact that $|\mathcal{Z}_{5\alpha} h_1(x, \omega)|$ is 5α -periodic function one can represent the absolute values of the elements of the matrix in **(e)** as

$$\begin{pmatrix} |\mathcal{Z}_{5\alpha} h_1(x + \alpha, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x + 2\alpha, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x - 2\alpha, \omega)| \\ |\mathcal{Z}_{5\alpha} h_1(x - \frac{2\alpha}{3}, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x + \frac{\alpha}{3}, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x + \frac{4\alpha}{3}, \omega)| \\ |\mathcal{Z}_{5\alpha} h_1(x - \frac{7\alpha}{3}, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x - \frac{4\alpha}{3}, \omega)| & |\mathcal{Z}_{5\alpha} h_1(x - \frac{\alpha}{3}, \omega)| \end{pmatrix}.$$

- Using Lemma 2 one can replace all elements, except those with indices $(1, 2)$, $(1, 3)$ and $(3, 1)$ with the main term of the corresponding series. We use an upper bound for the remaining (non-diagonal) elements replacing $\mathcal{Z}_{5\alpha}h_1(x - \frac{7\alpha}{3}, \omega)$ and $\mathcal{Z}_{5\alpha}h_1(x \pm 2\alpha, \omega)$ with respectively $3h_1(x - \frac{7\alpha}{3})$ and $3h_1(x \pm 2\alpha)$. Namely, for x near $\frac{5\alpha}{2}$ Lemma 2 gives

$$|\mathcal{Z}_{5\alpha}h_1(x, \omega)| = |h_1(x) - \mathcal{Z}_{5\alpha}h_1(x, \omega) - h_1(x)| \leq h_1(x) + C_{5\alpha}h_1(5\alpha - |x|).$$

Since the constant $C_{5\alpha} \approx 2$ we use the estimate $|\mathcal{Z}_{5\alpha}h_1(x, \omega)| \leq 3h_1(x)$ for x near $\frac{5\alpha}{2}$. Since we plan to use Theorem B we use an upper estimate for the non-diagonal elements. For the other values of x the error is negligible.

We will show that the following matrix satisfies the conditions of Theorem B.

$$\left(H_i^j\right)_{i,j=1}^3 = \begin{pmatrix} |h_1(x + \alpha)| & 3|h_1(x + 2\alpha)| & 3|h_1(x - 2\alpha)| \\ |h_1(x - \frac{2\alpha}{3})| & |h_1(x + \frac{\alpha}{3})| & |h_1(x + \frac{4\alpha}{3})| \\ 3|h_1(x - \frac{7\alpha}{3})| & |h_1(x - \frac{4\alpha}{3})| & |h_1(x - \frac{\alpha}{3})| \end{pmatrix}.$$

- For $0 \leq x \leq \frac{\alpha}{12}$ and $\sqrt{\frac{3}{5}} \leq \alpha \leq 1$ condition (ii) in Theorem B can now be verified by direct inspection.
- For $0 \leq x \leq \frac{\alpha}{12}$ and $\alpha \geq 1$ we consider the difference

$$H_i^i - \sum_{k, k \neq i} H_i^k, \quad 1 \leq i \leq 3.$$

If this expression is positive then condition (i) in Theorem B is met. Consider first the case $i = 1$. Then we need to verify

$$H_1^1 - H_1^2 - H_1^3 > 0. \quad (15)$$

Let $\alpha y = x$ for $0 \leq y \leq \frac{1}{12}$. Then (15) is equivalent to

$$\begin{aligned} & h_1(\alpha(y + 1)) - 3h_1(\alpha(y + 2)) - 3h_1(\alpha(2 - y)) \\ & > h_1(\alpha(\frac{1}{12} + 1)) - 3h_1(2\alpha) - 3h_1(\alpha(2 - \frac{1}{12})) \\ & > h_1(\frac{13\alpha}{12}) - 6h_1(\frac{23\alpha}{12}) = h_1(\frac{13\alpha}{12}) \left(1 - 6\frac{23}{13}e^{-\frac{5\pi\alpha^2}{2}}\right) \end{aligned}$$

which clearly is positive for all values $\alpha \geq 1$. The proof of the cases $i = 2$ and $i = 3$ follows along the same lines.

f. The case $1/12 \leq x \leq 1/6$ and $\alpha \geq \sqrt{\frac{3}{5}}$ can be treated similarly to the previous case. In this case one can verify that the submatrix of $\mathcal{P}_1(x, \omega)$ corresponding to the columns $t = 0, 2, 3$ fulfill condition (i) of Theorem B.

g. We prove that the case $\frac{\alpha}{6} \leq x \leq \frac{\alpha}{3}$ can be reduced to the previous ones. By substituting $x = \frac{\alpha}{3} - y$ we have

$$\begin{aligned} |\mathcal{Z}_{5\alpha} h_1(\frac{\alpha}{3} - y + \alpha t + r \frac{5\alpha}{3}, \omega)| &= |\mathcal{Z}_{5\alpha} h_1(\frac{\alpha}{3} - y + \alpha t + \alpha(r-1) \frac{5}{3} + \frac{5\alpha}{3}, \omega)| \\ &= |\mathcal{Z}_{5\alpha} h_1(y - \alpha(t+2) - \alpha(r-1) \frac{5}{3}, \omega)|. \end{aligned}$$

It is clear that the two cases are - up to permutations of rows/columns - similar for $0 \leq x \leq \frac{\alpha}{6}$ and $\frac{\alpha}{6} \leq x \leq \frac{\alpha}{3}$. \square

6 Conjecture

Calculations of the previous section become too cumbersome for arbitrary α, β with $\alpha\beta \in \mathbb{Q}$, $\alpha\beta < 1$. Numerous numerical calculations make us to believe that the following statement is true:

Conjecture *Let $\alpha\beta < 1$ and $\alpha\beta \neq (n-1)/n$, $n = 1, 2, \dots$. Then $\mathcal{G}(h_1, \alpha, \beta)$ in a frame in $L^2(\mathbb{R})$.*

Unfortunately the authors are at the moment not able to prove/disprove this statement even in the case $\alpha\beta \in \mathbb{Q}$.

The following figure suggest that $\alpha\beta = \frac{n-1}{n}$ are the only exceptional cases. Figure 1 shows the minimal eigenvalue of $\mathcal{P}(x, \omega)\mathcal{P}^T(x, \omega)$ for $0 \leq x \leq \frac{\alpha}{2p}$ and $0 \leq \omega \leq \frac{1}{\alpha}$ for $\alpha\beta = \frac{n-j}{n}$ for $5 \leq n \leq 201$ and $1 \leq j \leq n-1$. With this choice of n and j we have $\alpha\beta < 0.995$. In the experiments we have considered $\alpha = 1$.

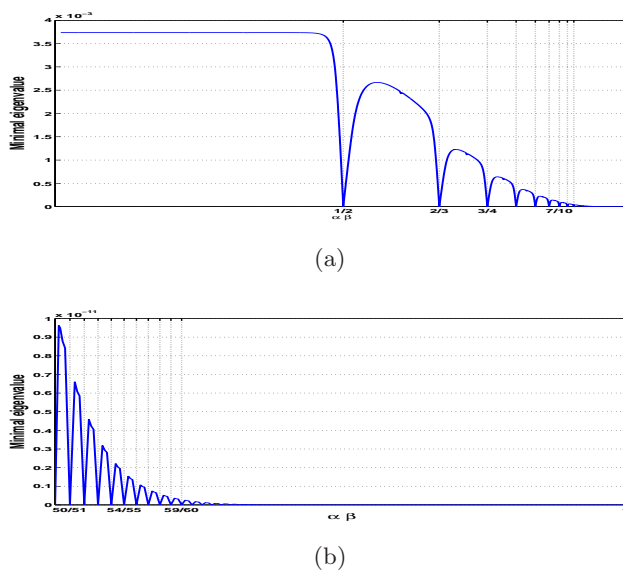


Figure 1: Minimal eigenvalue of $\mathcal{P}\mathcal{P}^T$ for (a) $\alpha\beta < 0.98$ and (b) $0.98 < \alpha\beta < 0.995$. In the experiments we have used $\alpha = 1$. Note that the scaling of the y-axis differs.

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